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On the Regularity of an Obstacle Control Problem[1]

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An optimal control problem of the obstacle for an elliptic variational inequality is considered, in which the obstacle is regarded as the control. To get the regularity of the optimal pair, a new related control problem is introduced. By proving the existence of an optimal pair to such a new control problem, the regularity of the optimal pair to the original problem is obtained. It turns out that the regularity obtained is sharp in general. Some other interesting properties of the optimal pair are also established. © 2001 Academic Press

Key Words: regularity; obstacle; control.

1. INTRODUCTION

In this paper, we consider the following optimal control problem:

Problem (C). Find a $\overline{\psi} \in H_0^1(\Omega)$, such that

$$J(\overline{\psi}) = \inf_{\psi \in H_0^1(\Omega)} J(\psi), \tag{1.1}$$

where

$$J(\psi) = \frac{1}{2}\int_\Omega \left\{ \left| T(\psi) - z \right|^2 + |\nabla\psi|^2 \right\} dx, \tag{1.2}$$

$\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{1,1}$ boundary $\partial\Omega$, $z \in L^2(\Omega)$ is a given target profile, and $y = T(\psi)$ is the solution of the following varia-

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tional inequality:

$$\begin{cases} y \in \mathbb{K}(\psi) = \left\{ \varphi \in H_0^1(\Omega) \mid \varphi \geq \psi, \text{a.e. } \Omega \right\}, \\ \displaystyle\int_\Omega \nabla y \cdot \nabla(\varphi - y)\, dx \geq 0, \qquad \forall \varphi \in \mathbb{K}(\psi). \end{cases} \tag{1.3}$$

It is well known that y satisfies (1.3) if and only if y is the minimizer of the functional $\varphi \mapsto \int_\Omega |\nabla\varphi|^2\, dx$ over the set $\mathbb{K}(\psi)$. Moreover, if y satisfies (1.3), then $y \in \mathscr{H}^+ \triangleq \{\varphi \in H_0^1(\Omega) \mid \int_\Omega \nabla\varphi \cdot \nabla v\, dx \geq 0, \forall v \in H_0^1(\Omega), v \geq 0$, a.e. $\Omega\}$ (*cf*. [15], *for example*).

In [1], Adams *et al*. introduced the above Problem (C). It was shown that there exists a unique optimal pair $(\bar{y}, \bar{\psi})$ to Problem (C) and $\bar{y}$ must be equal to $\bar{\psi}$. Moreover, if $z \in L^\infty(\Omega)$ (or $z \in C^\beta(\overline{\Omega})$ for some $\beta > 0$), then $\bar{y}$ is of class $C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ for any $\alpha \in (0,1)$ (or $C^{1,1}(\Omega)$).

The main purpose of this paper is to obtain further regularity and characterization of the optimal pair. More precisely, we show that $W^{2,p}$-regularity of the optimal pair to Problem (C) in the case that the target profile $z(\cdot)$ is only an $L^p(\Omega)$ $(2 \leq p < +\infty)$ function. When $z \in L^\infty(\Omega)$ or $z \in C^\beta(\overline{\Omega})$ for some $\beta \in (0,1)$, the $C^{1,\alpha}$ or $C^{1,1}$ boundary regularity of the optimal state is also obtained.

With the aid of the results on regularity and characterization of the optimal pair, some examples are constructed to reveal that such regularity is best possible in general. We also obtain some other interesting properties about the optimal pair and give some examples to show how to calculate the optimal pair by using the results in this paper.

Our main idea of getting the regularity of the optimal pair is to establish the existence theorem for a new related optimal control problem. We now explain this in detail.

Since the optimal pair $(\bar{y}, \bar{\psi})$ satisfies $\bar{y} = \bar{\psi} \in \mathscr{H}^+$, and for any $\varphi \in \mathscr{H}^+$, we have $T(\varphi) = \varphi$ (cf. [1]), thus finding the optimal pair $(\bar{y}, \bar{\psi})$ to Problem (C) is equivalent to finding the minimizer $\bar{y}$ such that

$$\tilde{J}(\bar{y}) = \inf_{y \in \mathscr{H}^+} \tilde{J}(y),$$

with

$$\tilde{J}(y) = \tfrac{1}{2}\int_\Omega \left\{ |y - z|^2 + |\nabla y|^2 \right\} dx.$$

Now we introduce the following optimal control problem:

Problem (C*). Find a $\bar{u} \in L^2_+(\Omega) = \{v \in L^2(\Omega) \mid v \geq 0, \text{ a.e. } \Omega\}$ such that

$$J^*(\bar{u}) = \inf_{u \in L^2_+(\Omega)} J^*(u),$$

where

$$J^*(u) = \frac{1}{2}\int_\Omega \{|y-z|^2 + uy\}\,dx, \tag{1.4}$$

and $y(\cdot) = y(\cdot\,; u(\cdot))$ is the solution to the equation

$$\begin{cases} -\Delta y = u, & \text{in } \Omega, \\ y|_{\partial\Omega} = 0. \end{cases} \tag{1.5}$$

The following result reveals the relation between Problem (C) and Problem (C*).

PROPOSITION 1.1. *If* $(\bar{y}, \bar{u})$ *is an optimal pair of Problem* (C*), *then* $(\bar{y}, \bar{y})$ *is the optimal pair of Problem* (C). *Conversely, if* $(\bar{y}, \bar{y})$ *is an optimal pair of Problem* (C) *and* $\bar{y} \in H^2(\Omega)$, *then* $-\Delta\bar{y} \geq 0$, *and by setting* $\bar{u} \triangleq -\Delta\bar{y} \in L^2_+(\Omega)$, $(\bar{y}, \bar{u})$ *is an optimal pair of Problem* (C*).

Proof. For any $u \in L^2_+(\Omega)$, let $y(\cdot) = y(\cdot\,; u(\cdot))$ be the corresponding solution of (1.5). We have $y \in \mathscr{H}^+ \cap H^2(\Omega)$ and

$$J^*(u) = \frac{1}{2}\int_\Omega \{|y-z|^2 + uy\}\,dx = \frac{1}{2}\int_\Omega \{|y-z|^2 + |\nabla y|^2\}\,dx = \tilde{J}(y).$$

On the other hand, for any $y \in \mathscr{H}^+ \cap H^2(\Omega)$, we have $-\Delta y \geq 0$. Thus by defining $u = -\Delta y$, we see that $y(\cdot)$ is the unique solution of (1.5) corresponding to $u(\cdot)$. Then

$$\tilde{J}(y) = \frac{1}{2}\int_\Omega \{|y-z|^2 + |\nabla y|^2\}\,dx = \frac{1}{2}\int_\Omega \{|y-z|^2 + uy\}\,dx = J^*(u).$$

Now, suppose that $(\bar{y}, \bar{u})$ is an optimal pair of Problem (C*). By the definition of $(\bar{y}, \bar{u})$, $\bar{y}$ is the minimizer of $\tilde{J}(y)$ over $\mathscr{H}^+ \cap H^2(\Omega)$. Since $\mathscr{H}^+ \cap H^2(\Omega)$ is dense in $\mathscr{H}^+$, $\bar{y}$ must be the minimizer of $\tilde{J}(\cdot)$ over $\mathscr{H}^+$. Hence $(\bar{y}, \bar{y})$ is an optimal pair of Problem (C).

Similarly, suppose that $(\bar{y}, \bar{y})$ is an optimal pair of Problem (C) and $\bar{y} \in H^2(\Omega)$. Then we have $\bar{y} \in \mathscr{H}^+ \cap H^2(\Omega)$. Therefore, $\bar{y}$ is the minimizer of $\tilde{J}(y)$ over $\mathscr{H}^+ \cap H^2(\Omega)$. Let $\bar{u} = -\Delta\bar{y}$. Then $\bar{u} \in L^2_+(\Omega)$. So $(\bar{y}, \bar{u})$ is an optimal pair of Problem (C*). ∎

Since the optimal pair of Problem (C) uniquely exists, Proposition 1.1 tells us that the H^2-regularity of an optimal pair to Problem (C) is equivalent to the existence of an optimal pair to Problem (C*). In Problem (C), the relation between state and control is nonlinear. But in Problem (C*), the relation between state and control is linear, which makes it much easier to deal with.

The rest of the paper is organized as follows.

In Section 2, we will prove the existence of optimal pairs to Problem (C*) and give a characterization of the optimal control. In Section 3, we will use the result obtained in Section 2 to explore some further interesting properties of optimal pairs. Some nontrivial examples will also be presented.

For further information on optimal control problems for variational inequalities see [1, 2, 5–7, 9–13, 16–18, 20, 21], for examples.

2. EXISTENCE AND CHARACTERIZATION OF OPTIMAL PAIR

We first introduce some preliminary lemmas.

LEMMA 2.1. *Let C be a constant. If $\varphi \in W^{m,p}(\Omega)$, $p \geq 1$, $m \geq 1$, then*

$$\partial^{\alpha}\varphi(x) = 0, \qquad a.e.\ x \in \{\tilde{x} \in \Omega \mid \varphi(\tilde{x}) = C\}, \forall 1 \leq |\alpha| \leq m,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_i, $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Proof. If $m = 1$, then the result is standard (cf. [19]).

If $m > 1$, then $\partial^{\beta}\varphi \in W^{m-1,p}(\Omega)$, $\forall |\beta| = 1$. Thus we can easily get the result by induction. ∎

The following lemma is a special case of the so-called "strong maximum principle."

LEMMA 2.2. *Suppose $L = -\Delta + \alpha I$, $\alpha \geq 0$, $y \in H^1(\Omega)$ satisfies $Ly \leq 0$ in Ω.*

(i) *Suppose for some ball $B \subset\subset \Omega$, we have*

$$\sup_B y = \sup_\Omega y \geq 0.$$

Then y must be a constant in Ω.

(ii) *Suppose $y \in C^1(\overline{\Omega})$, $y = 0$ on $\partial\Omega$, and $y \not\equiv 0$. Let $x_0 \in \partial\Omega$ be such that $\partial\Omega$ satisfies an interior sphere condition at x_0. Then*

$$\frac{\partial y}{\partial \nu}(x_0) > 0,$$

where ν is the outward normal on $\partial\Omega$.

The proof of Lemma 2.2 can be found in [14]. Let us now present the following lemma which is related to Problem (C*).

LEMMA 2.3. *Suppose $z \in L^2(\Omega)$. Denote*

$$\mathscr{U}_{ad}[0,b] = \begin{cases} \{u : \Omega \mapsto [0,b] \mid u \text{ is measurable}\}, & \text{if } 0 < b < +\infty, \\ L^2_+(\Omega), & \text{if } b = +\infty. \end{cases}$$

(i) *Suppose $0 < b < +\infty$, then there exists a unique $\bar{u} \in \mathscr{U}_{ad}[0,b]$ such that*

$$J^*(\bar{u}) = \inf_{u \in \mathscr{U}_{ad}[0,b]} J^*(u).$$

Let $\bar{y}$ be the solution of Eq. (1.5) *with u replaced by $\bar{u}$. Then there exists a $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ such that*

$$-\Delta\varphi = z - \bar{y} - \bar{u}, \qquad \text{in } \Omega, \tag{2.1}$$

and

$$\bar{u}(x) = \begin{cases} 0, & \text{if } \varphi(x) < 0, \\ z(x) - \bar{y}(x), & \text{if } \varphi(x) = 0, \ a.e.\ \Omega. \\ b, & \text{if } \varphi(x) > 0, \end{cases} \tag{2.2}$$

(ii) *Suppose $b = +\infty$. If there exists a $\bar{u} \in \mathscr{U}_{ad}[0,b]$ such that*

$$J^*(\bar{u}) = \inf_{u \in \mathscr{U}_{ad}[0,b]} J^*(u),$$

then such a $\bar{u}$ is unique. Let $\bar{y}$ be the state corresponding to it. Then there exists a $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ such that (2.1) *holds and*

$$\varphi \le 0, \qquad a.e.\ \Omega, \tag{2.3}$$

$$\bar{u}(x) = \begin{cases} 0, & \text{if } \varphi(x) < 0, \\ z(x) - \bar{y}(x), & \text{if } \varphi(x) = 0, \end{cases} \qquad a.e.\ \Omega. \tag{2.4}$$

Proof. (i) Suppose $0 < b < +\infty$. Let $\{u_k\}$ be a minimizing sequence of $J^*(\cdot)$ over $L^2_+(\Omega)$. We have

$$\|u_k\|_{L^2(\Omega)} \le b|\Omega|^{1/2} \le C, \qquad \forall k \ge 1,$$

where $|\Omega|$ is the Lebesgue measure of Ω. Therefore

$$\|y_k\|_{H^2(\Omega)} \le C, \qquad \forall k \ge 1,$$

for some constant C, where y_k is the state corresponding to the control u_k. So we can suppose that

$$y_k \to \bar{y}, \qquad \text{weakly in } H^2(\Omega), \text{ strongly in } H_0^1(\Omega),$$

$$u_k \to \bar{u}, \qquad \text{weakly in } L^2(\Omega).$$

Thus

$$\tfrac{1}{2}\int_\Omega \left\{|\bar{y}-z|^2+\bar{u}\bar{y}\right\}dx = \lim_{k\to\infty}\tfrac{1}{2}\int_\Omega \left\{|y_k-z|^2+u_k y_k\right\}dx$$
$$= \inf_{u\in\mathscr{U}_{ad}[0,b]} J^*(u).$$

Since $\mathscr{U}_{ad}[0,b]$ is convex, $\bar{u}\in\mathscr{U}_{ad}[0,b]$. On the other hand, it is easy to see that $\bar{y}$ must be the state corresponding to $\bar{u}$. Hence, $J^*(\bar{u}) = \inf_{u\in\mathscr{U}_{ad}[0,b]}J^*(u)$.

Suppose that $\tilde{u}\in\mathscr{U}_{ad}[0,b]$ satisfies $J^*(\tilde{u})=J^*(\bar{u})$. Let $\tilde{y}$ be the state corresponding to $\tilde{u}$. Then $\frac{\tilde{u}+\bar{u}}{2}\in\mathscr{U}_{ad}[0,b]$, and

$$J^*\left(\frac{\tilde{u}+\bar{u}}{2}\right)+\frac{1}{2}\int_\Omega\left\{\left|\frac{\tilde{y}-\bar{y}}{2}\right|^2+\left|\frac{\nabla\tilde{y}-\nabla\bar{y}}{2}\right|^2\right\}dx$$
$$=\frac{1}{2}(J^*(\tilde{u})+J^*(\bar{u}))=J^*(\bar{u})\le J^*\left(\frac{\tilde{u}+\bar{u}}{2}\right).$$

Therefore $\tilde{y}=\bar{y}$, leading to $\tilde{u}=\bar{u}$, and we have the uniqueness.

Now, let $v\in L^\infty(\Omega)$ such that for any $\varepsilon\in(0,1)$, $u_\varepsilon=\bar{u}+\varepsilon v\in\mathscr{U}_{ad}[0,b]$. We have

$$0\le\frac{J^*(u_\varepsilon)-J^*(\bar{u})}{\varepsilon}=\int_\Omega\left\{Y\left(\bar{y}+\frac{\varepsilon}{2}Y-z\right)+\left(\bar{y}+\frac{\varepsilon}{2}Y\right)v\right\}dx,$$

where

$$\begin{cases}-\Delta Y=v, & \text{in }\Omega,\\ Y|_{\partial\Omega}=0.\end{cases}\tag{2.5}$$

Letting $\varepsilon\to0^+$, we have

$$0\le\int_\Omega\{Y(\bar{y}-z)+\bar{y}v\}\,dx.$$

Set

$$\begin{cases}-\Delta\varphi=z-\bar{y}-\bar{u}, & \text{in }\Omega,\\ \varphi|_{\partial\Omega}=0.\end{cases}\tag{2.6}$$

We have $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$, and

$$
\begin{aligned}
0 &\le \int_\Omega \{Y(\Delta\varphi - \bar{u}) + \bar{y}v\}\, dx \\
&= \int_\Omega \{Y(\Delta\varphi + \Delta\bar{y}) + \bar{y}v\}\, dx \\
&= \int_\Omega \{\Delta Y(\varphi + \bar{y}) + \bar{y}v\}\, dx \\
&= \int_\Omega -v\varphi\, dx. \qquad (2.7)
\end{aligned}
$$

Thus we have

$$
\begin{cases}
\varphi \le 0, & \text{a.e. } \{x \in \Omega \mid \bar{u}(x) = 0\}, \\
\varphi = 0, & \text{a.e. } \{x \in \Omega \mid 0 < \bar{u}(x) < b\}, \\
\varphi \ge 0, & \text{a.e. } \{x \in \Omega \mid \bar{u}(x) = b\}.
\end{cases} \qquad (2.8)
$$

Consequently,

$$
\bar{u} = 0, \qquad \text{a.e. } \{x \in \Omega \mid \varphi(x) < 0\}, \qquad (2.9)
$$

and

$$
\bar{u} = b, \qquad \text{a.e. } \{x \in \Omega \mid \varphi(x) > 0\}. \qquad (2.10)
$$

To get the value of $\bar{u}$ on the set $\{x \in \Omega \mid \varphi(x) = 0\}$, noting that $\varphi \in H^2(\Omega)$, and by Lemma 2.1, we have

$$
-\Delta\varphi = 0, \qquad \text{a.e. } \{x \in \Omega \mid \varphi(x) = 0\}. \qquad (2.11)
$$

Thus

$$
z - \bar{y} - \bar{u} = 0, \qquad \text{a.e. } \{x \in \Omega \mid \varphi(x) = 0\}. \qquad (2.12)
$$

Combining (2.12) with (2.9)–(2.10) we get (2.2).

(ii) Suppose $b = +\infty$ and there exists a $\bar{u} \in \mathscr{U}_{ad}[0, b]$ such that

$$
J^*(\bar{u}) = \inf_{u \in \mathscr{U}_{ad}[0,b]} J^*(u).
$$

Like in (i), we can get the uniqueness of $\bar{u}$ and find a $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ which satisfies (2.1) and (2.7). Since $\bar{u} + \varepsilon v \in \mathscr{U}_{ad}[0, b]$, $\forall \varepsilon \in (0, 1)$, $v \in L_+^\infty(\Omega) \triangleq \{v \in L^\infty(\Omega) \mid v \ge 0, \text{ a.e. } \Omega\}$, we have (2.3) by (2.7). Similarly, we have (2.9), (2.11)–(2.12), and (2.4) follows. ∎

We now would like to show that if φ lacks $W^{2,p}$-regularity, then it is quite difficult to determine $\bar{u}$ on the set $E_0 \triangleq \{x \in \Omega \mid \varphi(x) = 0\}$. Certainly, if the Lebesgue measure of E_0 were zero, then $\bar{u}$ would be a bang-bang control (see (2.2)). In [12], it was proved that E_0 really has zero measure in some contexts. But for Problem (C), as we will see below, the set E_0 usually has positive measure. Therefore, to determine $\bar{u}$ on E_0 is very important. By introducing an approximation problem, i.e., by restricting the admissible control to a "good" space, we are able to get the $W^{2,p}$-regularity of φ which will lead to a description of $\bar{u}$ on E_0 by Lemma 2.1. Because we have explicit expressions of optimal controls in approximation problems, the uniform $W^{2,p}$-boundedness of the optimal states can be obtained, which enable us to go further. Let us make this more precise.

LEMMA 2.4. *If $z \in L^\infty(\Omega)$, then Problem* (C*) *admits a unique optimal pair* $(\bar{y}, \bar{u})$ *and*

$$\|\bar{y}\|_{H^2(\Omega)} \leq C\|z\|_{L^2(\Omega)},$$

$$\|\bar{u}\|_{L^2(\Omega)} \leq \|z\|_{L^2(\Omega)}.$$

Proof. Let $k \geq \|z\|_{L^\infty(\Omega)}$. By Lemma 2.3, denote $\bar{u}_k$ to be the minimizer of $J^*(\cdot)$ over $\mathscr{U}_{ad}[0,k]$, and $\bar{y}_k$ to be the corresponding state. Then we have $\varphi_k \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$-\Delta\varphi_k = z - \bar{y}_k - \bar{u}_k, \qquad \text{in } \Omega,$$

and

$$\bar{u}_k(x) = \begin{cases} 0, & \text{if } \varphi_k(x) < 0, \\ z(x) - \bar{y}_k(x), & \text{if } \varphi_k = 0, \\ k, & \text{if } \varphi_k(x) > 0, \end{cases} \qquad \text{a.e. } \Omega.$$

Since $\bar{u}_k \geq 0$, so $\bar{y}_k \geq 0$ by the maximum principle for (1.5). Thus we have

$$-\Delta\varphi_k = z - \bar{y}_k - k \leq \|z\|_{L^\infty(\Omega)} - k \leq 0, \qquad \text{a.e. } \{x \in \Omega \mid \varphi_k(x) > 0\}.$$

On the other hand, since $\bar{u}_k \in L^\infty(\Omega)$, we have $\bar{y}_k \in W^{2,p}(\Omega)$ for any $1 < p < +\infty$. Thus $\bar{y}_k \in L^\infty(\Omega)$. Therefore $z - \bar{y}_k - \bar{u}_k \in L^\infty(\Omega)$, which implies $\varphi_k \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. Thus $\Omega_k \triangleq \{x \in \Omega \mid \varphi_k(x) > 0\}$ is an open subset of Ω. If the Lebesgue measure $|\Omega_k|$ of Ω_k is positive, then $\varphi_k|_{\partial\Omega_k} = 0$ and by the weak maximum principle (cf. [14]), we have $\varphi_k(x) \leq 0$, a.e., Ω_k. This contradicts the definition of Ω_k. Therefore, we have $|\Omega_k| = 0$, i.e., $\varphi_k \leq 0$, a.e. Ω. Consequently,

$$\bar{u}_k(x) = (z - \bar{y}_k)\chi_{\{\varphi_k = 0\}}, \qquad \text{a.e. } \Omega.$$

This yields $\bar{u}_k(x) \le \|z\|_{L^\infty(\Omega)}$, a.e. Ω. Similarly, we have $\bar{u}_j \le \|z\|_{L^\infty(\Omega)} \le k$, $\forall j \ge k$. Therefore $\bar{u}_j$ also minimizes $J^*(\cdot)$ over $\mathscr{U}_{ad}[0,k]$. Then by the uniqueness, we must have

$$\bar{u}_j = \bar{u}_k, \qquad \text{a.e. } \Omega$$

This means that $\bar{u} \triangleq \bar{u}_k$ minimizes $J^*(\cdot)$ over $\mathscr{U}_{ad}[0,j]$, $\forall j > \|z\|_{L^\infty(\Omega)}$. Consequently, $\bar{u}$ must minimize $J^*(\cdot)$ over $L^2_+(\Omega)$. Thus Lemma 2.4 follows from Lemma 2.3. ∎

Now we establish the existence theorem of Problem (C*), which also gives a characterization of the optimal pair.

THEOREM 2.5. *Suppose $z \in L^2(\Omega)$. Then Problem* (C*) (*and so does Problem* (C)) *admits a unique optimal state $\bar{y} \in \mathscr{H}^+ \cap H^2(\Omega)$, and there exists a $\overline{\varphi} \in H_0^1(\Omega) \cap H^2(\Omega)$ such that*

$$\overline{\varphi}(x) \le 0, \qquad a.e.\ \Omega, \tag{2.13}$$

$$-\Delta \bar{y} = (z - \bar{y})\chi_{\{\overline{\varphi} = 0\}}, \tag{2.14}$$

$$-\Delta \overline{\varphi} = (z - \bar{y})\chi_{\{\overline{\varphi} < 0\}}. \tag{2.15}$$

Moreover, the pair $(\bar{y}, \overline{\varphi}) \in (\mathscr{H}^+ \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega))$ satisfying (2.13)–(2.15) *is unique.*

Proof. Let

$$z_k(x) = \begin{cases} z(x), & \text{if } |z(x)| \le k, \\ 0, & \text{if } |z(x)| > k, \end{cases}$$

and $(\bar{y}_k, \bar{u}_k)$ be an optimal pair to Problem (C*) with z replaced by z_k. By Lemma 2.4 we have

$$\|\bar{y}_k\|_{H^2(\Omega)} \le C\|z_k\|_{L^2(\Omega)} \le C\|z\|_{L^2(\Omega)},$$

$$\|\bar{u}_k\|_{L^2(\Omega)} \le \|z_k\|_{L^2(\Omega)} \le \|z\|_{L^2(\Omega)}.$$

By choosing subsequences, if necessary, we have

$$\bar{y}_k \to \bar{y}, \qquad \text{weakly in } H^2(\Omega), \text{ strongly in } H_0^1(\Omega),$$

$$\bar{u}_k \to \bar{u}, \qquad \text{weakly in } L^2(\Omega).$$

Then

$$-\Delta \bar{y} = \bar{u}, \qquad \text{in } \Omega.$$

Since $\|z_k - z\|_{L^2(\Omega)} \to 0$, we have

$$J^*(\bar{u}) \le J^*(u), \qquad \forall u \in L^\infty_+(\Omega).$$

Then it is easy to see

$$J^*(\bar{u}) \le J^*(u), \qquad \forall u \in L^2_+(\Omega).$$

So we have the existence, and (2.13)–(2.15) follows from Lemma 2.3.

Suppose $(\tilde{y}, \tilde{u}) \in (\mathscr{H}^+ \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega))$ satisfies (2.13)–(2.15). We have

$$\begin{aligned}
&\int_\Omega \left\{ |\bar{y} - \tilde{y}|^2 + |\nabla \bar{y} - \nabla \tilde{y}|^2 \right\} dx \\
&\quad = \int_\Omega (\bar{y} - \tilde{y})\big(\bar{y} - \tilde{y} - \Delta \bar{y} + \Delta \tilde{y}\big)\, dx \\
&\quad = \int_\Omega (\bar{y} - \tilde{y})(\Delta \overline{\varphi} - \Delta \tilde{\varphi})\, dx \\
&\quad = \int_\Omega (\Delta \bar{y} - \Delta \tilde{y})(\overline{\varphi} - \tilde{\varphi})\, dx \\
&\quad = \int_\Omega -\big(\overline{\varphi}\, \Delta \tilde{y} + \tilde{\varphi}\, \Delta \bar{y}\big)\, dx \\
&\quad \le 0.
\end{aligned}$$

Hence, $\bar{y} = \tilde{y}$. Consequently, $\overline{\varphi} = \tilde{\varphi}$, proving the uniqueness. ∎

3. FURTHER PROPERTIES OF THE OPTIMAL STATE

In this section, we let $\bar{y}$ be an optimal state of Problem (C*) corresponding to $z \in L^2(\Omega)$, and $\overline{\varphi} \in H_0^1(\Omega) \cap H^2(\Omega)$ be a function satisfying (2.13)–(2.15) as in Theorem 2.5. We see that $\bar{y}$ is the optimal state to Problem (C). For any measurable function f in Ω, f^+ and f^- will be denoted the positive part and the negative part of f, respectively, i.e., $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

It is proved in [1] that if $z \in L^\infty(\Omega)$, then $\bar{y} \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ for any $\alpha \in (0,1)$. Moreover, if $z \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, then $\bar{y} \in W^{2,\infty}_{loc}(\Omega) = C^{1,1}(\Omega)$. We strengthen these results in the following theorem.

THEOREM 3.1. *Suppose $z \in L^2(\Omega)$.*

(i) *If $z^+ \in L^p(\Omega)$ for some $p \in [2, +\infty)$, then $\bar{y} \in W^{2,p}(\Omega)$ and*

$$\|\bar{y}\|_{W^{2,p}(\Omega)} \le C \|z^+\|_{L^p(\Omega)}.$$

Consequently, if $z^+ \in L^p(\Omega)$ *for some* $p \in (n, +\infty)$, *then* $\bar{y} \in C^{1,1-n/p}(\overline{\Omega})$. *In particular, if* $z^+ \in L^\infty(\Omega)$, *then* $\bar{y} \in C^{1,\alpha}(\overline{\Omega})$ *for any* $\alpha \in (0,1)$. *In the one-dimensional case, if* $z^+ \in L^\infty(\Omega)$, *then* $\bar{y} \in C^{1,1}(\overline{\Omega})$.

(ii) *If* $z \in C^\alpha(\overline{\Omega})$, *and* $\partial\Omega$ *is of class* $C^{2,\alpha}$ *for some* $\alpha \in (0,1)$, *then* $\bar{y} \in W^{2,\infty}(\Omega) = C^{1,1}(\overline{\Omega})$.

Proof. (i) This follows from Theorem 2.5 immediately.

(ii) For $L = -\Delta + \alpha I$, $\alpha \geq 0$, we denote $L^{-1}z$ to be the solution of the equation

$$\begin{cases} Lv = z, & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Let $\psi_1 = (-\Delta)^{-1}(z - \bar{y})$, $\psi_2 = (-\Delta)^{-1}(z^+)$. We have $\psi_1 = \bar{y} + \overline{\varphi}$ by Theorem 2.5. Since $z \in C^\alpha(\overline{\Omega})$, so $\bar{y} \in C^{1,\alpha}(\overline{\Omega})$ by (i). We have $z^+, z - \bar{y} \in C^\alpha(\overline{\Omega})$. Thus, $\psi_1, \psi_2 \in C^{2,\alpha}(\overline{\Omega}) \cap H_0^1(\Omega)$. By Theorem 2.5, we know that $\overline{\varphi} \leq 0$, a.e. Ω, and $-\Delta\bar{y} \leq z^+$, a.e. Ω. So $\psi_1 \leq \bar{y} \leq \psi_2$. Then it is easy to see that $\bar{y}$ is the solution of the following bilateral-obstacle problem:

$$\begin{cases} y \in \mathbb{K}(\psi_1, \psi_2) = \left\{ v \in H_0^1(\Omega) \mid \psi_1 < v \leq \psi_2, \text{ a.e. } \Omega \right\}, \\ \displaystyle\int_\Omega \nabla\bar{y} \cdot \nabla(v - \bar{y})\, dx \geq 0, \qquad \forall v \in \mathbb{K}(\psi_1, \psi_2). \end{cases}$$

By the result of [8], we have $\bar{y} \in W^{2,\infty}(\Omega) = C^{1,1}(\overline{\Omega})$. ∎

In some sense, the $C^{1,1}$-regularity of the optimal state $\bar{y}$ is the best possible result which we can obtain. Even if z is analytic on Ω, $\bar{y}$ may have no C^2-regularity. To illustrate this, we present an example.

EXAMPLE 1. Consider $\Omega = (-1,1)$, $z(x) = x^2$. Let $(\bar{y}, \bar{u})$ be the optimal pair of the corresponding Problem (C*), and $\overline{\varphi}$ be the function as in Theorem 2.5. Then $\bar{y} \in C^{1,1}[-1,1]$ by Theorem 3.1. Consequently, $\overline{\varphi} \in C^{1,1}[-1,1]$. Noting that z is even, $\bar{y}, \overline{\varphi}, \bar{u}$ must also be even by the uniqueness. Therefore $\bar{y}'(0) = 0$. Since $-\bar{y}'' \geq 0$, we can see that $\bar{y}$ is nondecreasing on $[-1,0]$ and nonincreasing on $[0,1]$. By Theorem 3.2 (ii) which will be established below, we have $\bar{y} \not\equiv 0$. So $\bar{y}(0) = \sup_{x \in [-1,1]} \bar{y}(x) > 0$. Thus there exists a unique $a \in (0,1)$ such that

$$z > \bar{y}, \qquad \text{in } (-1, -a) \cup (a, 1),$$

and

$$z < \bar{y}, \qquad \text{in } (-a, a).$$

By Theorem 3.2 (i), we have $(-a, a) \subseteq \{x \in \Omega \mid \bar{u}(x) = 0\}$.

Moreover, we have $\overline{\varphi}(-a) = \overline{\varphi}(a) < 0$. Otherwise, suppose $\overline{\varphi}(-a) = \overline{\varphi}(a) = 0$. Since $\overline{\varphi} \leq 0$ and $\overline{\varphi} \in C^{1,1}[-1,1]$, we have $\overline{\varphi}'(-a) = \overline{\varphi}'(a) = 0$. On the other hand, $-\overline{\varphi}'' = z - \bar{y} < 0$ in the set $(-a, a)$. By Lemma 2.2, we have $\overline{\varphi} \equiv 0$ on $[-a, a]$. This contradicts $-\overline{\varphi}'' = z - \bar{y} < 0$ in $(-a, a)$.

Similarly, we have $\overline{\varphi}(x) < 0, \forall x \in [-a, a]$. Thus, there exists a $b \in (a, 1]$ such that

$$\overline{\varphi}(-b) = \overline{\varphi}(b) = 0,$$

and

$$\overline{\varphi}(x) < 0, \qquad \forall x \in (-b, b).$$

Therefore

$$-\bar{y}''(x) = 0, \qquad \text{a.e. } (-b, b).$$

Since $\bar{y} \not\equiv 0$, $b \in (a, 1)$. By (2.15), we have

$$-\overline{\varphi}'' = (z(x) - \bar{y}(x))\chi_{\{\overline{\varphi} < 0\}} \geq (z(b) - \bar{y}(b))\chi_{\{\overline{\varphi} < 0\}} \geq 0, \text{a.e. } (b, 1).$$

Noting that $\overline{\varphi}(b) = \overline{\varphi}(1) = 0$, we have $\overline{\varphi} \geq 0$ in $(b, 1)$. Thus by (2.13),

$$\overline{\varphi} = 0, \qquad \text{in } (b, 1).$$

So

$$-\bar{y}''(x) = z(x) - \bar{y}(x) \geq z(b) - \bar{y}(b) > 0, \qquad \forall x \in (b, 1).$$

Since $-\bar{y}''(x) = 0$ for $x \in (0, b)$, we find that $\bar{y}''$ is not continuous at b. In fact, $\bar{y}''(b)$ does not exist. So $\bar{y}$ is not a $C^2(\Omega)$ function.

Now, we will go further to get the precise expression of the optimal state $\bar{y}$. We have seen that $\overline{\varphi}(x) < 0$ when $|x| < b$, and $\overline{\varphi}(x) = 0$ when $b \leq |x| \leq 1$. So we have

$$\begin{cases} -\bar{y}'' = (x^2 - \bar{y})\chi_{\{b \leq |x| \leq 1\}}, & \text{a.e. } [-1, 1], \\ \bar{y}(-1) = \bar{y}(1) = 0. \end{cases}$$

Since $\bar{y} \in C^{1,1}[-1, 1]$, we can solve the equation and get

$$\bar{y} = \begin{cases} C_1 e^x + C_2 e^{-x} + x^2 + 2, & -1 \leq x \leq -b, \\ C_3 & -b < x < b, \\ C_1 e^{-x} + C_2 e^x + x^2 + 2, & b \leq x \leq 1, \end{cases}$$

where

$$C_1 = -\frac{3e^{2b+1} - 2be^{b+2}}{3^{2b} + e^2},$$

$$C_2 = -\frac{2be^b + 3e}{e^{2b} + e^2},$$

$$C_3 = -\frac{2be^{2b} + 6e^{b+1} - 2be^2}{e^{2b} + e^2} + b^2 + 2.$$

The only thing we need to do now is to determine b. We have

$$-\overline{\varphi}'' = x^2 - C_3, \qquad -b < x < b.$$

So

$$\overline{\varphi} = \begin{cases} 0, & |x| \geq b, \\ \frac{1}{2}C_3 x^2 - \frac{1}{12}x^4 + C_4 + C_5 x, & |x| < b. \end{cases}$$

Noting that $\overline{\varphi} \in C^{1,1}[-1,1]$, we have $\overline{\varphi}'(-b) = \overline{\varphi}'(b) = 0$. Therefore

$$C_3 b - \tfrac{1}{3}b^3 + C_5 = -C_3 b + \tfrac{1}{3}b^3 + C_5 = 0.$$

So $C_3 = \frac{1}{3}b^2$. Hence

$$\left(\tfrac{1}{3}b^2 - b + 1\right)e^{2(b-1)} - 3e^{b-1} + \tfrac{1}{3}b^2 + b + 1 = 0.$$

Then we get $\bar{y}$. Numerical calculations give the following results:

$$\begin{cases} b = 0.4171371847\ldots, \\ C_1 = -0.9725819959\ldots, \\ C_2 = -0.9720136636\ldots, \\ C_3 = 0.0580011436\ldots. \end{cases}$$

Next, we will give some further interesting results about $\bar{y}$ below. Then we will construct another example to show that $z \in C(\overline{\Omega})$ is not sufficient to obtain the $C^{1,1}$-regularity of $\bar{y}$ when the space dimension $n \geq 2$.

THEOREM 3.2. *Suppose $z \in L^2(\Omega)$.*

(i) *There exists a measurable set $E \subseteq \{x \in \Omega \mid z(x) > \bar{y}(x)\}$ such that*

$$-\Delta \bar{y} = \chi_E \cdot (z - \bar{y})^+ = \chi_E \cdot (z^+ - \bar{y})^+,$$

where χ_E *is the characteristic function of* E.

(ii) $\bar{y} = 0 \Leftrightarrow (-\Delta)^{-1} z \le 0$.

(iii) $-\Delta \bar{y} = z - \bar{y} \Leftrightarrow (I - \Delta)^{-1} z \le z$.

(iv) *If* $\{x \mid z(x) > 0\}$ *has a positive measure, then*

$$-\Delta \bar{y} = (z - \bar{y})^{+} \Leftrightarrow -\Delta \bar{y} = z - \bar{y}.$$

Proof. (i) Noting that $-\Delta \bar{y} \ge 0$ and $\bar{y} \ge 0$, we get the result from Theorem 2.5.

(ii) $(\Rightarrow)$ Let $\bar{y} = 0$. Then $-\Delta \overline{\varphi} = z - \bar{y} - \bar{u} = z$, and so $(-\Delta)^{-1} z = \overline{\varphi} \le 0$.

$(\Leftarrow)$ Let $(-\Delta)^{-1} z \le 0$. We have

$$\begin{aligned}
2J(\bar{y}) = 2\tilde{J}(\bar{y}) &= \int_\Omega \left\{ |\bar{y} - z|^2 + |\nabla \bar{y}|^2 \right\} dx \\
&= \int_\Omega \left\{ \bar{y}^2 - 2\bar{y}z + z^2 - \bar{y}\,\Delta \bar{y} \right\} dx \\
&= \int_\Omega \left\{ \bar{y}^2 - 2\bar{y}z + z^2 + \bar{y}(z - \bar{y} + \Delta \overline{\varphi}) \right\} dx \\
&= \int_\Omega \left\{ -\bar{y}z + z^2 + \bar{y}\,\Delta \overline{\varphi} \right\} dx \\
&= \int_\Omega \left\{ -\bar{y}z + z^2 + \overline{\varphi}\,\Delta \bar{y} \right\} dx \\
&= \int_\Omega \left\{ -\bar{y}z + z^2 \right\} dx \\
&= \int_\Omega \left\{ \Delta \bar{y} \cdot (-\Delta)^{-1} z + z^2 \right\} dx \\
&\ge \int_\Omega z^2 \, dx.
\end{aligned}$$

Since $2J(0) = \int_\Omega z^2 \, dx$, we have $J(0) = J(\bar{y})$. Hence $\bar{y} = 0$.

(iii) $(\Rightarrow)$ Let $-\Delta \bar{y} = z - \bar{y}$. Then $\bar{y} = (I - \Delta)^{-1} z$, and since $-\Delta \bar{y} \ge 0$, we have $(I - \Delta)^{-1} z = \bar{y} = z + \Delta \bar{y} \le z$.

$(\Leftarrow)$ Let $z \ge (I - \Delta)^{-1} z$. We denote $Z = (I - \Delta)^{-1} z$. It is easy to see that Z minimizes $\tilde{J}(\cdot)$ over $H_0^1(\Omega)$. Since $\Delta Z = z - Z \ge 0$, therefore $Z \in \mathscr{H}^+$ and Z minimizes $\tilde{J}$ over $\mathscr{H}^+$. Hence $\bar{y} = Z$, or equivalently, $-\Delta \bar{y} = z - \bar{y}$.

(iv) ($\Rightarrow$) Let $-\Delta\bar{y} = (z-\bar{y})^+$. Since $-\Delta\bar{y} - \Delta\bar{\varphi} = z - \bar{y}$, we have $-\Delta\bar{\varphi} = -(z-\bar{y})^- \le 0$. We want to prove $(z-\bar{y})^- = 0$, a.e. Ω, or equivalently, $\bar{\varphi} = 0$, a.e. Ω. Otherwise, by Lemma 2.2 (i),we have $\bar{\varphi} < 0$, in Ω. This yields that $-\Delta\bar{y} = 0$. So $\bar{y} = 0$ and $z = \bar{y} - \Delta\bar{y} - \Delta\bar{\varphi} = -\Delta\bar{\varphi} \le 0$, contradicting our assumption. Thus we must have $\bar{\varphi} = 0$, a.e. Ω, and so $-\Delta\bar{y} = z - \bar{y}$.

($\Leftarrow$) Let $-\Delta\bar{y} = z - \bar{y}$. Since $-\Delta\bar{y} \ge 0$, so $z \ge \bar{y}$ and we have $(z-\bar{y})^+ = z - \bar{y}$. Therefore $-\Delta\bar{y} = (z-\bar{y})^+$. ∎

It is proved in [1] that if $\omega \in H_0^1(\Omega)$ is a weak solution of the equation

$$\begin{cases} -\Delta\omega = (z-\omega)^+, & \text{in } \Omega, \\ \omega|_{\partial\Omega} = 0, \end{cases}$$

and such that $\omega \le z$, a.e. Ω, then $\bar{y} = \omega$. Since these conditions are equivalent to $(I-\Delta)^{-1}z \le z$, we see that this result is equivalent to the result $(I-\Delta)^{-1}z \le z \Rightarrow -\Delta\bar{y} = z - \bar{y}$.

By Theorem 3.2 (i), $-\Delta\bar{y} = 0$ on $supp(z^-)$. It seems that $\bar{y}$ depends only on z^+. In fact $\bar{y}$ is not determined by z^+. For example, let Ω be the unit ball $B_1(0)$, $z(x) = |x|^2 - M$. Then

$$(-\Delta)^{-1}z(x) = \frac{1}{4(n+2)}(1-|x|^4) - \frac{M}{2n}(1-|x|^2).$$

Let $1 > M \ge n/(n+2)$; we have $(-\Delta)^{-1}z \le 0$. Thus, corresponding to z, the optimal state $\bar{y}(\cdot; z) \equiv 0$. On the other hand, by Lemma 2.3, we can easily see that $(-\Delta)^{-1}z^+(x) > 0$ in Ω. Consequently, by Theorem 3.2 (ii), the optimal state $\bar{y}(\cdot; z^+)$ corresponding to z^+ is not 0.

As is shown in [1], in two simple cases, we can get $\bar{y}$ easily.

Case 1. If $z \le 0$, then $\bar{y} = 0$.

Case 2. If $-\Delta z \ge 0$ and $z|_{\partial\Omega} \ge 0$, then we have $\bar{y} = (I-\Delta)^{-1}z$.

To see this, compare

$$\begin{cases} -\Delta z + z \ge z, & \text{in } \Omega, \\ z|_{\partial\Omega} \ge 0, \end{cases}$$

with

$$\begin{cases} -\Delta Z + Z = z, & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, \end{cases}$$

and we have $(I-\Delta)^{-1}z \le z$ by the weak maximum principle (cf. [14]). Therefore by Theorem 3.2 (iii), we have $\bar{y} = (I-\Delta)^{-1}z$.

Moreover, we have the following corollary.

COROLLARY 3.3. *There exists a* $\theta = \theta(\Omega) \in (0,1)$ *such that if*

$$\inf_{x\in\Omega} z(x) \geq \theta \sup_{x\in\Omega} z(x),$$

then $\bar{y} = (I - \Delta)^{-1} z$. *In particular, when* $\Omega = B_r$, *the ball of radius r in* $\mathbb{R}^n$ *centered at the origin, we can choose* $\theta = 1 - n/(n + e^{r^2/2})$.

Proof. Let $Z = (I - \Delta)^{-1}\chi_\Omega$, i.e., $Z \in H_0^1(\Omega)$ and $-\Delta Z + Z = 1$, in Ω. We have $Z \in C^\infty(\overline{\Omega})$. By the weak maximum principle, $Z \geq 0$, in Ω. Denote $\theta_0 \triangleq \sup_\Omega Z$. It is easy to see that $\theta_0 > 0$. We want to prove $\theta_0 < 1$.

Let $Z(x_0) = \theta_0$. Then $x_0 \in \Omega$, and we have $-\Delta Z(x_0) \geq 0$. So

$$Z(x_0) = 1 + \Delta Z(x_0) \leq 1.$$

Thus

$$-\Delta Z(x) = 1 - Z(x) \geq 1 - Z(x_0) \geq 0, \qquad \forall x \in \Omega.$$

We claim that $-\Delta Z(x_0) > 0$. Otherwise, suppose that $-\Delta Z(x_0) = 0$. Then denote $W = \Delta Z$; we have

$$\begin{cases} -\Delta W + W = 0, & \text{in } \Omega, \\ W|_{\partial\Omega} = -1, \end{cases}$$

and $\sup_\Omega W \leq 0 = W(x_0)$. So $\sup_\Omega W = 0$. By Lemma 2.2 (i), we see that W must be constant in Ω. This is a contradiction. Consequently, $\theta_0 < 1$.

Now, suppose $\theta \in (\theta_0, 1)$, $\inf_{x\in\Omega} z(x) \geq \theta \sup_{x\in\Omega} z(x)$. Then $z \geq 0$, in Ω, and we have

$$(I - \Delta)^{-1} z(x) \leq \Big(\sup_\Omega z\Big)(I - \Delta)^{-1}\chi_\Omega(x) \leq \theta \sup_\Omega z \leq \inf_\Omega z \leq z(x).$$

So we have $\bar{y} = (I - \Delta)^{-1} z$ by Theorem 3.2 (iii).

When $\Omega = B_r$, we have $Z(x) = h(|x|)$, where $h \in C^\infty[0,r]$ and satisfies

$$\begin{cases} -h''(s) - \dfrac{n-1}{s} h'(s) + h(s) = 1, & \text{in } (0, r], \\ h(r) = 0,\ h'(0) = 0. \end{cases}$$

Moreover, we have

$$h(s) = \frac{M}{M+1} - \frac{1}{M+1}\left(\frac{1}{n\cdot 2}s^2 + \frac{1}{n(n+2)\cdot 2\cdot 4}s^4 + \frac{1}{n(n+2)(n+4)\cdot 2\cdot 4\cdot 6}s^6 + \cdots\right)$$

with

$$M = \frac{1}{n\cdot 2}r^2 + \frac{1}{n(n+2)\cdot 2\cdot 4}r^4 + \frac{1}{n(n+2)(n+4)\cdot 2\cdot 4\cdot 6}r^6 + \cdots$$
$$\leq \frac{1}{n}\left\{1 + \frac{r^2}{2} + \frac{1}{2!}\left(\frac{r^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{r^2}{2}\right)^3 + \cdots\right\} = \frac{1}{n}e^{r^2/2}.$$

So

$$\theta_0 = \max_{s\in[0,r]} h(s) = h(0) = \frac{M}{M+1} \leq 1 - \frac{n}{n+e^{r^2/2}},$$

and we can choose $\theta = 1 - n/(n + e^{r^2/2})$. ∎

As an application, we give another example to calculate $\bar{y}$.

EXAMPLE 2. Let $n = 1$, $\Omega = (0,1)$, $z = 4 + x\sin(2k\pi x)$, where k is a given integer. Then we see that

$$\inf_{x\in(0,1)} z(x) \geq 3 \geq 5\left(1 - \frac{1}{1+e^{1/8}}\right) \geq \left(1 - \frac{1}{1+e^{1/8}}\right)\sup_{x\in(0,1)} z(x).$$

Therefore

$$-\bar{y}'' = \bar{y} = 4 + x\sin(2k\pi x).$$

Hence, we get

$$\bar{y}(x) = \frac{4k\pi}{(4k^2\pi^2+1)^2}\cos(2k\pi x) + \frac{1}{4k^2\pi^2+1}\sin(2k\pi x) + 4 - \left(\frac{4k\pi}{(4k^2\pi^2+1)^2} + 4\right)\frac{e^x + e^{1-x}}{e+1}.$$

Now let us give another example to show that in high dimension, $z \in C(\overline{\Omega})$ is not sufficient for the $C^{1,1}$-regularity of $\bar{y}$.

EXAMPLE 3. For simplicity, we suppose $n = 2$.

Let $\Omega = B_{1/4}(0)$. Choose $\xi(\cdot) \in C_0^\infty(\Omega)$ such that $\xi(x) = 1$ for $|x| < \frac{1}{8}$. Let $x = (x_1, x_2) \in \Omega$,

$$f(x) = f(x_1, x_2) = \xi(x_1, x_2)\left(x_1^2 - x_2^2\right) \ln\left|\ln\left(x_1^2 + x_2^2\right)\right|.$$

Then it is easy to check that $f \notin C^{1,1}(\Omega)$ and $f \in W_0^{2,p}(\Omega)$ for any $p \in [1, +\infty)$. On the other hand, f is C^∞ smooth except at the origin. Near the origin, for $x = (x_1, x_2) \in B_{1/8}(0)$, we have

$$\begin{aligned} f_{x_1 x_1}(x_1, x_2) = {} & 2\ln\left|\ln\left(x_1^2 + x_2^2\right)\right| + \frac{10x_1^2 - 2x_2^2}{\left(x_1^2 + x_2^2\right)\ln\left(x_1^2 + x_2^2\right)} \\ & - \frac{4x_1^2\left(x_1^2 - x_2^2\right)}{\left(x_1^2 + x_2^2\right)^2 \ln\left(x_1^2 + x_2^2\right)} - \frac{4x_1^2\left(x_1^2 - x_2^2\right)}{\left(x_1^2 + x_2^2\right)^2 \ln^2\left(x_1^2 + x_2^2\right)}, \\ f_{x_2 x_2}(x_1, x_2) = {} & -2\ln\left|\ln\left(x_1^2 + x_2^2\right)\right| - \frac{10x_2^2 - 2x_1^2}{\left(x_1^2 + x_2^2\right)\ln\left(x_1^2 + x_2^2\right)} \\ & + \frac{4x_2^2\left(x_2^2 - x_1^2\right)}{\left(x_1^2 + x_2^2\right)^2 \ln\left(x_1^2 + x_2^2\right)} + \frac{4x_2^2\left(x_2^2 - x_1^2\right)}{\left(x_1^2 + x_2^2\right)^2 \ln^2\left(x_1^2 + x_2^2\right)}, \\ -\Delta f(x_1, x_2) = {} & -\frac{8\left(x_1^2 - x_2^2\right)}{\left(x_1^2 + x_2^2\right)\ln\left(x_1^2 + x_2^2\right)} + \frac{4\left(x_1^2 - x_2^2\right)}{\left(x_1^2 + x_2^2\right)\ln^2\left(x_1^2 + x_2^2\right)}. \end{aligned}$$

So $v \triangleq -\Delta f \in C(\overline{\Omega})$. Now let $z = M + v$ with M a large number such that

$$M + \inf_{x \in \Omega} v(r) \geq \left(1 - \frac{2}{2 + e^{1/32}}\right)\left(M + \sup_{x \in \Omega} v(x)\right).$$

By Corollary 3.3, we have $-\Delta \bar{y} = z - \bar{y} = M - \bar{y} + v$. Therefore $\bar{y} = (-\Delta)^{-1}(M - \bar{y}) + f$. Since $\bar{y} \in C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$, we have $(-\Delta)^{-1}(M - \bar{y}) \in C^{2,\alpha}(\overline{\Omega})$. Since $f \notin C^{1,1}(\Omega)$, $\bar{y} \notin C^{1,1}(\Omega)$.

In the proof of Theorem 3.1 (ii), we have seen that if $\bar{y}$ is the optimal state of Problem (C), then it is a solution of variational inequality (1.3) for some ψ. The following theorem shows the converse is also true in some sense.

THEOREM 3.4. *Suppose* $\psi \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, $2 \leq p < +\infty$, $y = T(\psi)$ *is the solution of variational inequality* (1.3). *Then* y *is an optimal state of Problem* (C) *with* $z = y - \Delta\psi$. *In particular, we have*

$$-\Delta y = -\Delta\psi\, \chi_{\{y = \psi\}}, \qquad a.e.\ \Omega.$$

Proof. If we use the fact $y \in W^{2,p}(\Omega)$ (cf [3, 4, 15]), we can get the result easily by Lemma 2.1 and Theorem 2.5. But we would like to give a proof here without using the fact $y \in W^{2,p}(\Omega)$.

Now, let $z = y - \Delta\psi$. Since $y \in H_0^1(\Omega)$ and $\psi \in W^{2,p}(\Omega) \subset H^2(\Omega)$, we have $z \in L^2(\Omega)$. By Theorem 2.5, there exists a $(\bar{y}, \overline{\varphi}) \in (\mathscr{H}^+ \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega))$ satisfying (2.5). Thus we need only prove $y = \bar{y}$.

By the definition of y, we have

$$\int_\Omega \nabla y \cdot \nabla(v - y)\, dx \geq 0, \qquad \forall v \in \mathbb{K}(\psi). \tag{3.1}$$

Modifying the proof of Theorem 2.5 and noting that $z = y - \Delta\psi$, $z - \bar{y} = -\Delta\bar{y} - \Delta\overline{\varphi}$, and $\psi - \overline{\varphi} \in \mathbb{K}(\psi)$, we have

$$\begin{aligned}
&\int_\Omega \left\{ |\bar{y} - y|^2 + |\nabla\bar{y} - \nabla y|^2 \right\} dx \\
&\quad = \int_\Omega \left\{ (\bar{y} - y)^2 + \nabla\bar{y} \cdot (\nabla\bar{y} - 2\nabla y) + |\nabla y|^2 \right\} dx \\
&\quad = \int_\Omega \left\{ (\bar{y} - y)^2 + \Delta\bar{y} \cdot (2y - \bar{y}) + |\nabla y|^2 \right\} dx \\
&\quad = \int_\Omega \left\{ (\bar{y} - y)(\bar{y} - y - \Delta\bar{y}) + y\,\Delta\bar{y} + |\nabla y|^2 \right\} dx \\
&\quad = \int_\Omega \left\{ (\bar{y} - y)(\Delta\overline{\varphi} - \Delta\psi) + y\,\Delta\bar{y} + |\nabla y|^2 \right\} dx \\
&\quad = \int_\Omega \left\{ \bar{y}\,\Delta\overline{\varphi} + y\,\Delta(\psi - \overline{\varphi}) - \bar{y}\,\Delta\psi + y\,\Delta\bar{y} + |\nabla y|^2 \right\} dx \\
&\quad = \int_\Omega \left\{ \overline{\varphi}\,\Delta\bar{y} + (y - \psi)\,\Delta\bar{y} + y\,\Delta(\psi - \overline{\varphi}) + |\nabla y|^2 \right\} dx \\
&\quad = \int_\Omega \left\{ (y - \psi)\,\Delta\bar{y} - \nabla y \cdot \nabla(\psi - \overline{\varphi} - y) \right\} dx \\
&\quad \leq 0.
\end{aligned}$$

Hence $\bar{y} = y$. ∎

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